Brain Memory Working

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PINN-PAD: Physics Informed Neural Networks in Padova - Dipartimento di Ingegneria Civile, Edile e Ambientale

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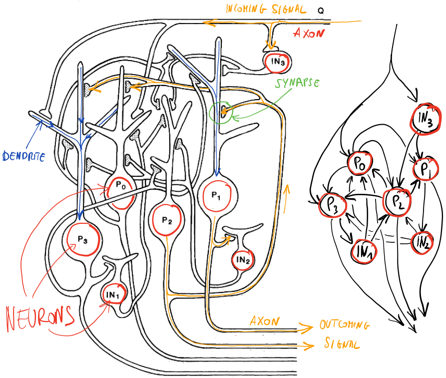
HOPFIELD MODEL OF NEURAL NETWORK

Original discrete Hopfield model with N neurons. At the n-th time step activation potential at neuron i:

$$V_i^{(n)}, \qquad i = 1, \ldots, N.$$

T_{ij} = conductance between neurons i and j.
Potentials updating rule:

$$V_i^{(n+1)} = g\left(\sum_{j=1}^{N} T_{ij} V_j^{(n)}\right), \quad g(u) := \begin{cases} +1 & u \geqslant a, \\ -1 & u < a. \end{cases}$$



A neuronal circuit.

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HOPFIELD MODEL OF NEURAL NETWORK

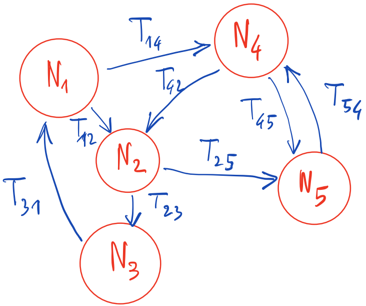
The associated continuous dynamics

$$\dot{u}_i = \sum_{j=1}^{N} T_{ij} g(u_j) - u_i$$

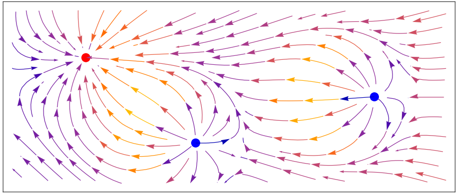
where $g(x)$ is a sigmoidal activation function, is described by the vector field

$$X_i(u) = \sum_{j=1}^{N} T_{ij} g(u_j) - u_i.$$

Neuronal network.



Possible associated continuous dynamics X.



Blue dots: sources. Red dots: sinks.

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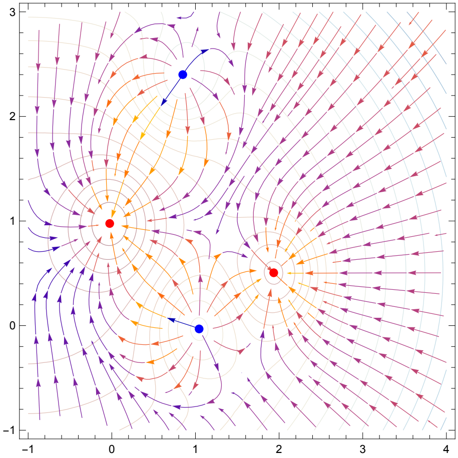
Symmetric Hopfield model of neural network

- Symmetry: $T_{ij} = T_{ji}$
- Symmetry + Constancy : Energy landscape

$$E(V) := -\frac{1}{2} T_{ij} V_i V_j + \sum_{i=1}^{N} \int_0^{V_i} g^{-1}(x)\, dx.$$

- Gradient dynamics $X = -\nabla E$
- Dynamics drives the potential pattern $(V_1, \ldots, V_N)$ towards the local energy minimum.



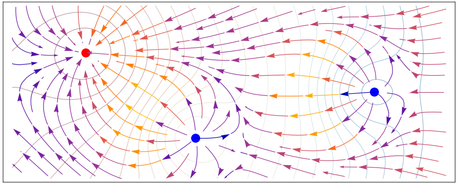
Contour plot of the energy landscape. Gradient-type dynamics. Minima = red dots. Maxima = blue dots.

Network updates

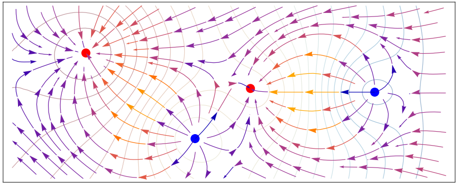
- Hebbian updates are discontinuos and can only add new patterns until saturation.

$$T_{ij}^{new} = T_{ij}^{old} + \frac{1}{N}\widehat{V_i}\widehat{V_j}$$

- Excessively rigid updating scheme: the network is forced to learn a pattern.



Hebbian updates to T_{ij} add new patterns/landscape minima.



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Krotov: non constant but symmetric

Krotov extension:

- the interaction matrix T_{ij} varies with electric potential V_i according to this "unusual" rule:

$$T_{ij} \to \boxed{T_{ij}(V) := \frac{\partial^2 \Phi}{\partial V_i \partial V_j}(V),} \qquad \text{(still symmetric)} \qquad (\spadesuit)$$

- $T_{ij}(V)$ is the Hessian of the Lagrangian $\Phi(V)$.
- via Legendre transform we obtain the Hamiltonian energy:

$$E(V) := -\left(\nabla \Phi(V) \cdot V - \Phi(V) - \sum_{i=1}^{N} \int_0^{V_i} g^{-1}(x)\, dx \right).$$

- new gradient vector field:

$$\widehat{X}_i(V) := -\nabla_i E(V) = \nabla^2_{ij} \Phi(V) \cdot V_j - g^{-1}(V_i).$$

We have decripted condition (♠):

Theorem

In a simply connected domain, the closure condition:

$$\left(T_{kj,i} - T_{ki,j}\right) V_k = 0, \qquad (\star)$$

is equivalent to the gradient structure for $\widehat{X}$ and to the existence of a Lyapunov-like energy function.

Remark

Under the stronger condition:

$$T_{kj,i} - T_{ki,j} = 0, \qquad (\diamondsuit)$$

we gain the Krotov hypothesis (♠): $T = \nabla^2\Phi$, but not to $(\star)$.

First symmetric non constant proposal

In the more general condition $(\star)$, we define now the corresponding Energy function. Let :

$$W(x) := \int_0^1 T_{ij}(\lambda x)\lambda x_i x_j d\lambda,$$

we set

$$E(V) := -W(V) + \sum_{i=1}^{N} \int_0^{V_i} g^{-1}(\lambda)\, d\lambda,$$

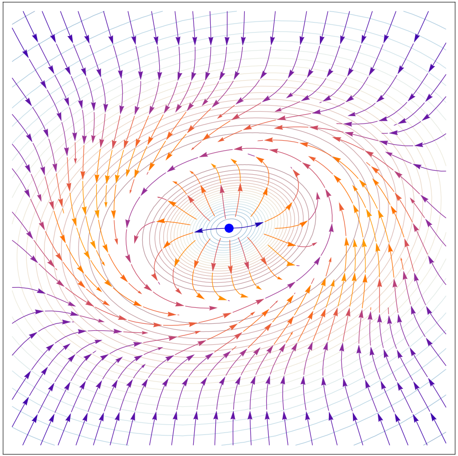
and obtain

$$\underbrace{\widehat{X_i} = -\nabla E(V)}_{\textit{gradient-like dynamics}} = T_{ij}(V)\, V_j - g^{-1}(V_i).$$

Need for breaking the symmetry

- Physiology states that T_{ij} is asymmetric: connections are directed, i.e., specific structures are dedicated to outgoing (axons) and incoming (dendrites) connections.
- Features non comprised by symmetric interactions:
  - oscillations / memory association,
  - wandering (instability),
  - forgetting and recovering memories.

  Oscillations and instability need for asymmetry in T_{ij}



Oscillations or limit cycles are only possible with asymmetry.

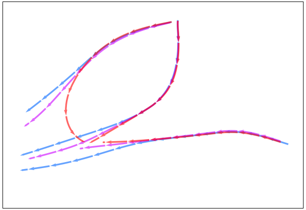
A new proposal: asymmetric optimal control

- Starting constant matrix A_{ij} non symmetric.
- ξ-controlled adjustments:

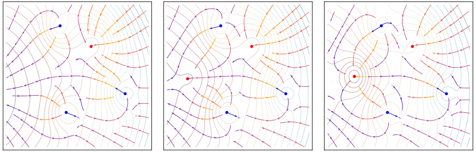
$$\boxed{T_{ij}(\xi) := A_{ij} + \xi_{ij}}, \qquad |\xi_{ij}| \leqslant K$$

- ξ-controlled Hopfield dynamics:

$$\dot{u}_i(t) = X_i\left(u(t), \xi(t)\right) = \\ = \sum_{j=1}^{N} \left(A_{ij} + \xi_{ij}(t)\right) g(u_j(t)) - u_i(t).$$



Trajectories dynamically evolving during motion.



Dynamical evolution of the energy landscape (symmetric).

Ideas already appeared for instance in

G. Parisi, Asymmetric neural networks and the process of learning, J.Phys A, 1986

D. Vardalaki et al, Filopodia are a structural substrate for silent synapses in adult neocortex. Nature 2022

A new proposal: asymmetric optimal control

Remark on sparsity:

- If N is large, A_{ij} is sparse.
- ξ_{ij} may update only $A_{ij} \neq 0$, or...
- ξ_{ij} may also act on $A_{ij} = 0$, lighting up
  - existing but silent synapses
  - build brand new synapses not existing before

Proposal: fix $0 < k \ll K$:

- if $A_{ij} \neq 0 \implies |\xi_{ij}(t)| \leqslant K,$
  i.e., if a connection A_{ij} between neurons i and j already exists, then the corresponding update may be "strong": $\xi_{ij} \leqslant K$.
- if $A_{ij} = 0 \implies |\xi_{ij}(t)| \leqslant k \ll K,$
  i.e., if A_{ij} is silent, then only smaller updates are possible $\xi_{ij} < k \ll K$.

  - *D. Vardalaki et al, Filopodia are a structural substrate for silent synapses in adult neocortex. Nature 2022*

Resuming the updating scheme we write: $|\xi_{ij}(t)| \leqslant (k, K)$.

A new proposal: Asymmetric Optimal Control

Surprisingly: there exist a perfectly fit powerful mathematical framework:

Infinite Horizon Optimal Control Problem.

- Differential Constraint:

$$\dot{u}_i(t) = X_i\,(u(t), \xi(t)) = \sum_{j=1}^{N} (A_{ij} + \xi_{ij}(t))\, g(u_j(t)) - u_i(t). \qquad (\dagger)$$

- $e^{-\lambda t}$-discounted variational principle:

$$\boxed{\min_{\xi(\cdot)} J\left(u^{(0)}, \xi(\cdot)\right) = \min_{\xi(\cdot)} \int_0^{+\infty} \underbrace{\left(\left|X(u(t, u^{(0)}, \xi(\cdot)), \xi(t))\right|^2 + |\xi(t)|^2\right)}_{\text{Lagrangian: } \ell(u,\xi)} e^{-\lambda t} dt}$$

where $u(u^{(0}, \xi(\cdot))$ solves the differential constraint $(\dagger)$.

Infinite Horizon Optimal Control Problem.

- The Lagrangian of the Control Problem:

$$\ell(u,\xi) = |X(u,\xi)|^2 + |\xi|^2 \,,$$

- $|X|^2$ small $\Rightarrow$ towards equilibra,
- $|\xi|^2$ small $\Rightarrow$ cheap solutions in terms of matrix modification.
- The discount $e^{-\lambda t}$ ensures convergence.
- Control problem: for fixed $u^{(0)}$ find the minimizing controls $\xi(\cdot)$:

$$\inf_{|\xi(t)|\leqslant(K,k)} J\left(u^{(0)}, \xi(\cdot)\right) \,,$$

DISCUSSION OF THE CONTROLLED MODEL

A controlled trajectory starting from the input pattern $u^{(0)}$ may fall in one of the following classes:

- reach existing equilibrium without activating the controls $\xi = 0$:

$$\lim_{t\to\infty} X(u(t, u^{(0)}, 0), 0) = X(u^\star, 0) = 0,$$

  i.e., the initial pattern $u^{(0)}$ has been recognized.
- The controls $\xi(t) \neq 0$ operate to minimize $J(u, \xi)$ and asymptotically drive to a new equilibrium:

$$\lim_{t\to\infty} X(u(t, u^{(0)}, \xi(t)), \xi(t)) = X(u^{\star\star}, \xi^{\star\star}_\infty) = 0,$$

  i.e., the initial pattern $u^{(0)}$ has been recorded in the network $T_{ij} \to T_{ij} + \xi^{\star\star}_\infty$ and a new equilibrium $u^{\star\star}$ has been created.
- ...

Discussion of the controlled model

- ...
- Assume that $\bar{u}$ is an equilibrium for the synaptic matrix T_{ij}. A sequence of alterations to T_{ij} are operated:

$$T_{ij} \to T_{ij} + \xi_\infty^\alpha + \cdots + \xi_\infty^\omega, \qquad |\xi_\infty^\alpha + \cdots + \xi_\infty^\omega| > K.$$

  In the new configuration the pattern $\bar{u}$ cannot be recognized, i.e., the pattern $\bar{u}$ has been forgot.
- (continued) Successive alteration ξ_∞^η may bring back the synaptic network closer to the starting configuration:

$$|\xi_\infty^\alpha + \cdots + \xi_\infty^\omega + \xi_\infty^\eta| \leqslant K,$$

  allowing to recover the old equilibrium $\bar{u}$, i.e., a memory has been restored.
- ...

- ...
- Given the asymmetry of T_{ij}, limit cycles are possibly approached ($\xi = 0$) or created ($\xi_\infty \neq 0$) during the controlled motion:

$$\lim_{t\to\infty} \text{dist}\left(u(t, u^{(0)}, \xi(t)), \mathcal{U}\right) = 0, \qquad \mathcal{U} \subseteq \mathbb{R}^N \quad \text{(limit cycle)},$$

Instability with oscillations: this situation can be interpreted as memory association.

- *H Yan et al., Nonequilibrium Landscape Theory of Neural Networks, PNAS 2013*

- Controls are activated during the motion ($\xi(t) \neq 0$) but they are not able to reach or create any equilibrium:

$$\lim_{t\to\infty} u(t, u^{(0)}, \xi(t)) \qquad \text{does not exist.}$$

Instability with wandering: pattern not found nor created.

FURTHER DISCUSSION / CONCLUSIONS

- Final Value Theorem
- Hamilton-Jacobi-Bellman Equation
- Dynamic Programming Principle
- Pareto optimization: conservative/innovative attitudes:

$$J_\mu\left(u^{(0)},\xi(\cdot)\right) := \int_0^{+\infty} \underbrace{\left((1-\mu)\left|X(u(t,u^{(0)},\xi(\cdot)),\xi(t))\right|^2 + \mu\,|\xi(t)|^2\right)}_{\text{Lagrangian: } \ell_\mu(u,\xi)} e^{-\lambda t}dt$$

  - if $0 < \mu \ll 1$ large values of ξ are allowed, letting the network explore innovative configurations,
  - if $0 \ll \mu < 1$ large values of ξ are penalized and the network is more prone towards existing minima: conservative attitude.

Thanks for your attention!

Brain memory working. Optimal control behavior for improved Hopfield-like models
https://arxiv.org/abs/2305.14360

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